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The Two-Stream Instability Studied with Four One-Dimensional Plasma Simulation Models

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Four one-dimensional plasma simulation models have been compared with regard to the electrostatic two-stream instability. The primary reason for making these comparisons was to determine the extent to which physical results depend on numerical method for a problem in which collective effects dominate. Previously, Lewis, Sykes, and Wesson compared these four simulation models using a stable double-streaming situation as a test problem. In that case the comparisons were with regard to collisional effects, energy conservation, and momentum conservation; however, because a stable test problem was used, only tentative conclusions could be drawn as to the comparison among the models when they are applied to a problem in which collective effects dominate. We have applied the models to compute the evolution of a two-stream instability and compared the time dependence of the electric energy as determined by each of the models. The models are characterized by the representation of the electric potential and by how the electric field is computed from the potential; both linear and quadratic splines are used to represent the potential or field. The major result of our comparisons is that the evolution of the electric energy of a two-stream unstable plasma does not depend strongly on the choice of model. There is a much stronger dependence on the random numbers that are chosen to represent the initial distribution function in phase space.

1. INTRODUCTION

Particle plasma simulation has been applied to a wide variety of linear and nonlinear problems in recent years. The basic electrostatic models used have been largely of the nearest-grid-point or particle-in-cell variety. In these models the self-consistent electric field derived from the electrostatic potential is either a

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constant or linear function between mesh points. It is the purpose of this paper to determine the effect on plasma behavior of piecewise quadratic representations of the field and/or potential.

The models used here were first compared by Lewis, Sykes and Wesson [1]. In a one-dimensional plasma simulation of a Maxwellian distribution of positrons streaming stably through a Maxwellian distribution of electrons, these authors found that the models were very similar with regard to collisional effects. It was suggested, however, that this surprising negative result might not apply to problems in which collective effects dominate. We study such a problem here, namely, the instability of two equal oppositely streaming warm electron beams. This instability provides us with a well-documented test case. Rosenbluth [2] has analyzed the linear regime; and the simulations of Roberts and Berk [3], Morse and Nielson [4], and Chu, Gula, and Mason [5] demonstrate the qualitative aspects of the nonlinear evolution of this instability.

In this paper we first briefly discuss the four numerical simulation models which we used in our study of the electron two-stream instability. Test runs are then summarized for the four models using three different randomly chosen sets of initial conditions. Diagnostic measurements of momentum and energy conservation, and electric field energy versus time are presented. Our results confirm conclusions reached by Lewis, Sykes, and Wesson [1]; no significant differences between the four models are observed.

2. The Models and Test Problem

The four models used are designated by M/N, where M and N are the degrees of the piecewise polynomials used to represent the potential and electric field, respectively. M or N may be 0 (piecewise constant), 1 (continuous piecewise linear), or 2 (smooth piecewise quadratic). With this nomenclature, the four models investigated were 1/0, 1/1, 2/1, and 2/2. For example, the 2/1 model has a smooth piecewise quadratic potential with a continuous piecewise linear field. The 1/1 model corresponds to the standard PIC scheme [6].

For our test runs a fixed, positive, neutralizing background was employed. Periodic boundary conditions were used, and particles were advanced by the standard time-centered leapfrog scheme. For completeness, we describe the models briefly here; they are described in detail in Ref. 1. The potential $\phi(x, t)$ is defined with respect to a spatial grid consisting of N cells of equal length in a period,and the potential is required to vanish at the endpoints of the grid. The representation of $\phi(x, t)$ is

$$\phi(x, t) = \sum_{n=1}^{N-1} \alpha_n(t)\, g_n(x), \tag{1}$$

where the functions $g_n(x)$ are dimensionless local basis functions for first- or second-degree splines that vanish at the endpoints. The coefficients $\alpha_n(t)$ have the dimension of potential.

Poisson's equation is represented as

$$\sum_m T_{nm}\alpha_m = -\rho_n/\epsilon_0 , \tag{2}$$

where the T matrix is a variational representation of d^2/dx^2, ρ_n represents the charge density assigned to the nth grid point, and ϵ_0 is the permittivity of free space. T_{nm} and ρ_n are defined by

$$T_{nm} = -\int_{x_0}^{x_N} g_n'(x)\, g_m'(x)\, dx, \tag{3}$$

where x_0 and x_N are the positions of the left and right endpoints, and

$$\rho_n = \sum_i Q_i g_n(y_i), \tag{4}$$

where Q_i and y_i are the charge and position of the ith particle. The electric field $E(x, t)$ for the models is

$$E(x, t) = -\sum_{n=1}^{N-1} \alpha_n(t)\, g_n'(x) \qquad \text{(1/0 and 2/1 models)} \tag{5}$$

or

$$E(x, t) = \sum_{n=1}^{N} \sigma_n(t)\, h_n(x) \qquad \text{(1/1 and 2/2 models);} \tag{6}$$

the coefficients σ_n and α_n are related by

$$\sum_{n=1}^{N} \sigma_n h_n(y_k) = -\sum_{n=1}^{N-1} \alpha_n g_n'(y_k) \qquad (k = 1, 2,..., N). \tag{7}$$

The functions $h_n(x)$ and $g_n(x)$ are identical except near the endpoints; the functions $h_n(x)$ are local basis functions for periodic first- or second-degree splines. The positions y_k coincide with grid points for the 1/1 model and are midway between grid points for model 2/2.

The electric field energy U for these models is $-(\epsilon_0/2) \sum_{i,j} \alpha_i T_{ij} \alpha_j$, where U in the continuum case is defined by

$$U = -\frac{\epsilon_0}{2} \int \phi(x, t) \frac{d^2\phi}{dx^2}\, dx. \tag{8}$$

For the test runs whose results are presented in the next section, the initial loading consisted of two counterstreaming warm electron beams, each beam containing $4992 = 78 \times 64$ particles. The initial velocities in each beam were chosen randomly from a Maxwellian distribution. The thermal speeds of the two Maxwellian distributions were equal, and the mean velocities were ± 3 times the thermal speed of either beam. Initially, the particles were spaced equally along the coordinate axis, the particles of one beam alternating with those of the other beam. The total number of cells, N, was 64, and the initial spacing between adjacent particles was $[64/(2 \times 4992)]$ times the cell length. Each endpoint of the grid was half that distance from the nearest particle. The cell length was one Debye length, where the Debye length corresponds to the initial thermal speed of either beam. The time step used for advancing the particles with the standard leapfrog scheme was $0.04\ \tau_p$, where τ_p is the plasma period corresponding to the total average particle density. The computations for the results presented were performed using a CDC 7600 computer.

3. Simulation Results

In Fig. 1 results on the evolution of the electric field energy are summarized for the four models up to time $16\tau_p$. The three columns in the figure correspond to three distinct random number sets used for the velocity initialization. The field energy plots display the usual linear growth, saturation, and subsequent oscillation as found by previous investigators [3–5]. Pronounced differences are seen in these curves for a given M/N model from one random number set to another.[1] For any particular velocity initialization, all four models show strikingly similar behavior up to about time $8\tau_p$. For longer times, the general behavior (i.e., oscillations) is similar with only small detailed differences being evident.

The fluctuation in total energy, ΔE, and the fluctuation in momentum, ΔP, were calculated for the time period $16\tau_p$; they are presented in Table I for each case. ΔE and ΔP are defined as follows:

$$\Delta E = \frac{E_{\max} - E_{\min}}{E_{\max} + E_{\min}},$$

$$\Delta P = \left| \frac{P_{\max} - P_{\min}}{N_0 m v_{\mathrm{th}}} \right|,$$

where N_0 is the total number of particles, m is the electron mass, v_{th} is the initial

[1] It is worthwhile to note that care was taken in assuring that the respective random number sets used were completely different. These correspond to the first, second, and third group of $12 \times N_0$ random numbers generated by the CDC Library subroutine RANF.

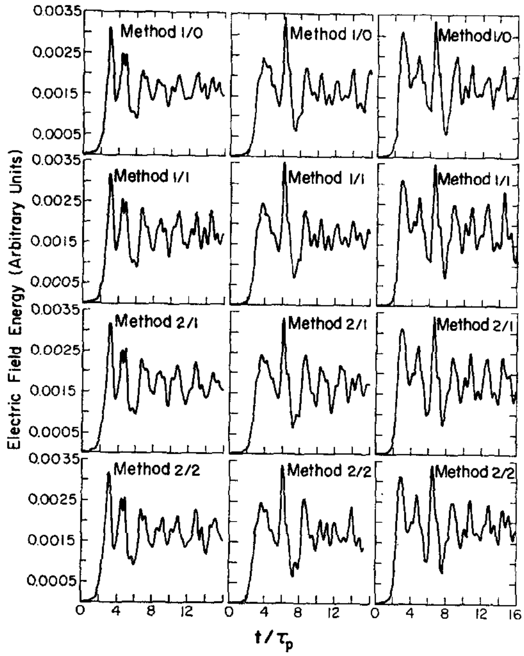

FIG. 1. Electric field energy versus time in units of the plasma period τ_p for models 1/0, 1/1, 2/1, and 2/2. The three columns of subfigures correspond to the three different velocity initializations that were used.

TABLE I

Energy and Momentum Fluctuations for all Four Models and the Three Velocity Initializations

Model		1/0	1/1	2/1	2/2
1st set of random numbers	$\Delta E =$	1.5×10^{-3}	8.5×10^{-4}	1.2×10^{-3}	6.7×10^{-4}
	$\Delta P =$	3.9×10^{-3}	7.2×10^{-12}	1.0×10^{-3}	3.1×10^{-11}
2nd set	$\Delta E =$	1.9×10^{-3}	8.0×10^{-4}	1.0×10^{-3}	5.7×10^{-4}
	$\Delta P =$	3.3×10^{-3}	2.7×10^{-11}	9.1×10^{-4}	7.5×10^{-12}
3rd set	$\Delta E =$	1.2×10^{-3}	7.9×10^{-4}	1.2×10^{-3}	9.6×10^{-4}
	$\Delta P =$	3.0×10^{-3}	4.4×10^{-11}	9.4×10^{-4}	4.8×10^{-11}

thermal speed of either beam, and E_{max}, P_{max}, (E_{min}, P_{min}) are the maximum (minimum) total energy and momentum of the system over the total time $16\tau_p$. Energy is conserved to at least 0.2% for all models. Momentum is conserved to at least 0.4% for the 1/0 and 2/1 models, while the 1/1 and 2/2 models conserve momentum to machine roundoff.

4. Conclusions

We have studied the development of the two-stream instability using four numerical simulation models. For a given set of initial velocities, the histories of electric field energy agree well with one another for all four models up to time $8\tau_p$. While differences in the field energy graphs are evident for later times, the overall impression is that all models behaved in essentially the same way. A strong dependence on initialization was displayed by all models. Total energy and momentum were conserved well for each model and run; the 2/2 and 1/1 models exhibited momentum conservation to machine roundoff.

On the basis of our results with this problem, there appears to be little difference between the four models in their applicability to problems in which very active collective motions of the plasma are present. The additional smoothing provided by the 2/1 and 2/2 models had little effect. Because the models that use a piecewise linear potential are mathematically simpler and computationally more economical, we conclude that those models are generally preferable for one-dimensional electrostatic plasma simulations.

Although the results of our attempt to find an improved simulation method for one-dimensional collisionless plasmas are pessimistic, it should be borne in mind that simulation is useful for determining some gross features of the evolution of plasmas. Conclusions similar to those reported here for one-dimensional simulation have been made recently for two-dimensional simulation on the basis of a comparison of three two-dimensional simulation models [7].

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